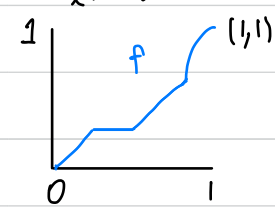
Math 564: Real analysis and measure theory
Lecture 13

Measure isomorphism theorem. Every atomless Borel probability measure μ on a Polish space X is isomorphic to $([0,1], \lambda)$. In fact, there is a Borel isomorphism $f : X \to [0,1]$ with $f_* \mu = \lambda$.

Proof. Because X is atomless, each singleton is μ-null, so X must be unctbl since otherwise $\mu(X) = 0$. By the Borel isomorphism theorem, there is a Borel isomorphism $g : X \to [0,1]$, so by replacing X with $[0,1]$ and μ with $g_* \mu$, we may assume that $X = [0,1]$ and μ is a atomless Borel prob. meas. on $[0,1]$. E.g. take $\mu|_{[0,\frac{1}{2}]} \equiv 0$ and $\mu|_{(\frac{1}{2},1]} \equiv 2 \cdot \lambda|_{(\frac{1}{2},1]}$.

Let $f : [0,1] \to [0,1]$ be defined by $x \mapsto \mu([0,x])$. This is an increasing (maybe not strictly) and continuous; indeed it is right continuous because of downward monotone convergence $\left(\mu([0,x]) = \lim_{x_n \to x^+} \mu([0,x_n])\right)$ and left continuous because of upward monotone convergence and atomlessness $\left(\mu([0,x]) = \mu([0,x)) = \lim_{x_n \to x^-} \mu([0,x_n])\right)$. Furthermore, $f^{-1}([0,y]) = [0,x]$ where $\mu([0,x]) = y$ and this x is maximum such, so

$$\lambda([0,y]) = y = \mu\left(f^{-1}([0,y])\right) = f_* \mu([0,y]).$$



Since the sets $[0,y]$ generate the Borel σ-alg, the measures λ and $f_* \mu$ coincide. It remains to show that f is a bijection on a conull set. f is not bijective because there might be intervals $[a,b]$ on which f is constant, but then $\mu((a,b]) = 0$ and there are only ctbly many such maximal intervals because these are pairwise disjoint and $[0,1]$ is separable. So the union Z of all these maximal intervals $(a,b]$ is still μ-null and $f|_{X \setminus Z} : X \setminus Z \to [0,1]$ is a bijection. $\square$

Cor. Every σ-finite Borel measure μ on a Polish space X is isomorphic to $(\mathbb{R}, \lambda)$.

Proof. Write X as a ctbl disjoint union of finite positive measure pieces and use that each is isomorphic to an interval in $\mathbb{R}$. Details left as an exercise. $\square$

Def. A measure space $(X, \mathcal{B}, \mu)$ is called *standard* if μ is σ-finite and $(X, \mathcal{B})$ is standard Borel.

Remark. In dynamics and probability theory, one mainly works with standard probability space (since we know that the atomless ones are all isomorphic).

We can restate:

Theorem. A standard atomless infinite measure space is isomorphic to $(\mathbb{R}, \lambda)$ and a standard atomless prob. measure space is isomorphic to $([0,1], \lambda)$.

Integration

Given a measurable space $(X, \mathcal{B})$, we denote

$$L := L(X, \mathcal{B}) := \text{the set of all } \mathcal{B}\text{-measurable functions } X \to \overline{\mathbb{R}} := [-\infty, \infty].$$

$$L^+ := L^+(X, \mathcal{B}) := \{f \in L(X, \mathcal{B}) : f \geq 0\}.$$

Note that $L(X, \mathcal{B})$ is a vector space and $L^+(X, \mathcal{B})$ is closed under non-negative linear combination. Both are closed under products and limits.

Def. An *integral* on $L^+(X, \mathcal{B})$ is a ctbly additive linear functional $I : L^+ \to [0, \infty]$, i.e.

(i) $I(\alpha \cdot f + \beta \cdot g) = \alpha \cdot I(f) + \beta \cdot I(g)$ for all $\alpha, \beta \geq 0$ and $f, g \in L^+$.

(ii) $I\left(\sum_{n \in \mathbb{N}} f_n\right) = \sum_{n \in \mathbb{N}} I(f_n)$.

Observation. Every integral I on $L^+(X, \mathcal{B})$ defines a measure μ_I on $(X, \mathcal{B})$ by:

$$\mu_I(B) := I(\mathbb{1}_B).$$

Proof. Indeed, $\mu_I(\emptyset) = I(\mathbb{1}_\emptyset) = I(0) = 0$ and if $B = \bigsqcup_{n \in \mathbb{N}} B_n$, then $\mathbb{1}_B = \sum_{n \in \mathbb{N}} \mathbb{1}_{B_n}$, so

$$\mu_I(B) = I(\mathbb{1}_B) = I\left(\sum_{n \in \mathbb{N}} \mathbb{1}_{B_n}\right) = \sum_{n \in \mathbb{N}} I(\mathbb{1}_{B_n}) = \sum_{n \in \mathbb{N}} \mu_I(B_n).$$

$\square$

We would like to solve the inverse problem, indeed we will prove:

Theorem. Every measure μ on $(X, \mathcal{B})$ admits an unique integral $I: \mathcal{L}^+(X, \mathcal{B}) \to [0, \infty]$ s.t. $\mu_I = \mu$.
This I is called the integral over μ, denoted $\int f \, d\mu := \int f(x) \, d\mu(x) := I(f)$.

In the remaining time we will prove this theorem by gradually defining this I.

Remark. Once I is defined on $\mathcal{L}^+$, we will be able to also define it on a subspace of $L(X, \mathcal{B})$ using the fact that for each $f \in L(X, \mathcal{B})$,

$$f = f_+ - f_-,$$

where $f_-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{o.w.} \end{cases}$ and $f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{o.w.} \end{cases}$

Def. A function $f \in L(X, \mathcal{B})$ is called simple if it is a finite linear combination of indicator functions of sets from $\mathcal{B}$. So nonnegative simple functions are the nonnegative linear combinations of indicator functions. Denote by $S(X, \mathcal{B})$ (resp. $S^+(X, \mathcal{B})$) the subspace of simple (resp. nonnegative simple) functions.
Note that a nonnegative simple function f is of the form

$$f = \sum_{i<n} \alpha_i \mathbb{1}_{A_i}$$

for some $n \in \mathbb{N}$, $\alpha_i \geq 0$, $A_i \in \mathcal{B}$.
It has many representations but there is a standard one:

Observation. A function $f \in L(X, \mathcal{B})$ is simple $\iff$ $f(X)$ is finite.

Def. For a simple function f with $f(X) = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$, the representation

$$f = \sum_{i<n} \alpha_i \mathbb{1}_{f^{-1}(\alpha_i)}$$

is called standard.
In particular $X = \bigsqcup_{i<n} f^{-1}(\alpha_i)$.

Def. Given a measure μ on $(X, \mathcal{B})$, we define its integral on $S^+(X, \mathcal{B})$ by setting for each $f \in S^+(X, \mathcal{B})$,

$$(*) \qquad \int f \, d\mu := \sum_{i<n} \alpha_i \, \mu(A_i),$$

where $f = \sum_{i<n} \alpha_i \mathbb{1}_{A_i}$ is some representation of f. It is an exercise HW to show that this is well-defined (i.e. doesn't depend on the representation).

Remark. We could alternatively define the integral for the standard representation and then prove that $(*)$ holds for any representation.

Prop. The integral on S^+ satisfies the following:

(i) Linearity: $\int (\alpha f + \beta g) \, d\mu = \alpha \cdot \int f \, d\mu + \beta \int g \, d\mu$.

(ii) Nonnegativity: $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$. Equiv: $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(iii) $\int f \, d\mu = 0 \Rightarrow f = 0$ a.e.

(iv) Each $f \in S^+$ defines a measure μ_f on $(X, \mathcal{B})$ by

$$\mu_f(B) := \int (\mathbb{1}_B \cdot f) \, d\mu =: \int_B f \, d\mu.$$

Proof. (i)–(iii) follow by definition, and we prove (iv). (i) implies $\mu_f(\emptyset) = 0$, so we verify ctbl additivity. Let $B = \bigsqcup_{n \in \mathbb{N}} B_n$, so $\mathbb{1}_B = \sum_{n \in \mathbb{N}} \mathbb{1}_{B_n}$. Let $f = \sum_{i<m} \alpha_i \mathbb{1}_{A_i}$. Then:

$$\mu_f(B) = \int \mathbb{1}_B \cdot f \, d\mu = \int \sum \alpha_i \mathbb{1}_{A_i \cap B} \, d\mu = \sum_{i<m} \alpha_i \, \mu(A_i \cap B) = \sum_{i<m} \alpha_i \sum_{n \in \mathbb{N}} \mu(A_i \cap B_n)$$

$$= \sum_{n \in \mathbb{N}} \sum_{i<m} \alpha_i \, \mu(A_i \cap B_n) = \sum_{n \in \mathbb{N}} \int (\mathbb{1}_{B_n} \cdot f) \, d\mu = \sum_{n \in \mathbb{N}} \mu_f(B_n). \qquad \square$$